The Non-Measurable

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Properties of Measure

We know how to measure certain subsets of $\mathbb{R}^d$
(cubes, spheres, rectangles, etc.) In general, can we assign a unique value or "measure" $|E|$ for each set $E \subseteq \mathbb{R}^d$ such that certain "nice" properties hold?

(i) $0 \leq |E| \leq \infty$.

(ii) A unit cube $Q = [0,1]^d$ has a measure $|Q| = 1$.

(iii) **Countable Additivity:** Given a finitely or countably many disjoint subsets of $\mathbb{R}^d$, $(E_1, E_2, ...)$, then

$$\left| \bigcup_k E_k \right| = \sum_k |E_k|.$$

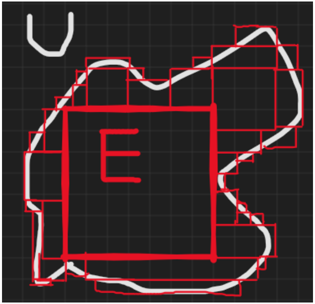
(iv) **Translation Invariant:** $|E + h| = |E|$ for all $h \in \mathbb{R}^d$.

Lebesgue Measure

As it turns out, the answer is *no*. This is a result from the Axiom of Choice. However, we can relax the condition that *every* subset of $\mathbb{R}^d$ is measurable, to get a measure that satisfies the four properties.

Definition

A set $E \subseteq \mathbb{R}^d$ is *Lebesgue measurable* if $\forall \varepsilon > 0, \exists$ open $U \supseteq E$ such that $|U \backslash E|_e \leq \varepsilon$.
*Where $|\cdot|_e$ represents an external *Lebesgue measurable*.



The Theorem

Theorem

There exists a set N that is not Lebesgue measurable. Taking it further, there exists no measure function $\mu : \mathcal{P}(\mathbb{R}) \to [0, \infty]$ that satisfy all the properties $(i - iv)$.

Proof idea.

- Construct a set N using the Axiom of Choice.
- Using the Steinhaus Theorem, show that N is not Lebesgue measurable.
- Prove by contradiction there exists no measure that satisfies all four properties for all subsets in $\mathbb{R}^d$.

□

Axiom of Choice

The construction of a non-measurable set requires the Axiom of Choice, defined below,

Axiom (Axiom of Choice)

Given a nonempty set S, let P be the family of all nonempty subsets of S, There exists a function $f : P \to S$ such that $f(A) \in A$ for each $A \in P$.

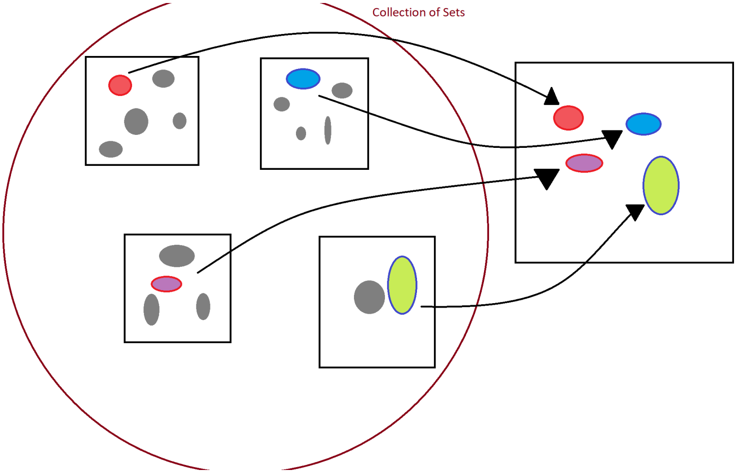
An equivalent statement is as follows.

Axiom (Axiom of Choice) *equivalent*

The Cartesian product $\prod_{i \in I} A_i$ of any collection $\{A_i\}_{i \in I}$ of nonempty sets is nonempty.

The latter statement implies that given a collection of nonempty disjoint sets $\{A_i\}_{i \in I}$, there exists a set N such that it contains exactly one element from each A_i.

Axiom of Choice



Constructing a Non-Measurable set

1. Let $\sim$ be an equivalence relation between two points $x, y \in \mathbb{R}$, such that
$$x \sim y \iff x - y \in \mathbb{Q}.$$
2. Moreover, let the the equivalence class of x be denoted as
$$[x] = \{y \in \mathbb{R} : x - y \in \mathbb{Q}\} = \{q + x : q \in \mathbb{Q}\} = \mathbb{Q} + x.$$
3. Each equivalence class of $\sim$ is a translation of the set of rational numbers by x. As consequence, there are an uncountable number of distinct $[x]$ that partition $\mathbb{R}$, each of which is a countable set.
4. Then by the **Axiom of Choice**, there is a set $N \subseteq \mathbb{R}$ that contains exactly one element from each distinct equivalence class of $\sim$.

Steinhaus Theorem

Theorem

If $E \subseteq \mathbb{R}$ is Lebesgue measurable and $|E| > 0$, then the set of differences

$$E - E = \{x - y : x, y \in E\}$$

contains the interval centered at 0.

Lemma

Given measurable subset E of $\mathbb{R}^d$ and $0 < \alpha < 1$, such that $0 < |E|_e < \infty$, there exists a cube Q such that $|E \cap Q|_e \geq \alpha |Q|$.

Steinhaus Theorem: Lemma

Lemma

Given measurable subset E of $\mathbb{R}^d$ and $0 < \alpha < 1$, such that $0 < |E|_e < \infty$, there exists a cube Q such that $|E \cap Q|_e \geq \alpha|Q|$.

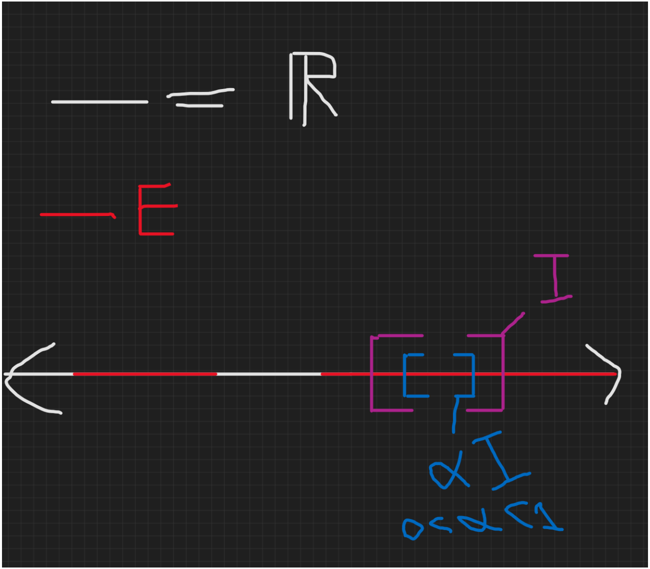
1. Let E be a measurable subset in $\mathbb{R}^d$, and let $\alpha \in (0, 1)$.
2. Let $Q = \bigcup_{i=1}^{n} Q_i$ be a collection of nonempty boxes of equal measure such that there is at least one $i \in \{1, ..., n\}$ such that $Q_i \subseteq E$.
3. Such an i exists as $|E|_e > 0$.
4. By scaling down $|Q_i|$ by α, we have,

$$|Q_i|_e \geq \alpha|Q_i|.$$

5. Thus by containment, there exists a cube Q_i, where

$$|E \cap Q_i|_e = |Q_i|_e \geq \alpha|Q_i|_e = \alpha|Q_i|.$$

Steinhaus Theorem: Lemma visual



Steinhaus Theorem Proof

Theorem

If $E \subseteq \mathbb{R}$ is Lebesgue measurable and $|E| > 0$, then the set of differences $E - E = \{x - y : x, y \in E\}$ contains the interval centered at 0.

Proof.

- By the Lemma, There exists an interval $I = [a, b]$ such that $|F| = |E \cap I| \geq \frac{3}{4}|I|$. Translating by t we have, $I \cap (I + t) = [a, b + t]$ if $t \geq 0$, and $I \cap (I + t) = [a - |t|, b]$ if $t \leq 0$, then

$$I \cup (I + t) \leq |I| + |t|.$$

- Consider the case where F and $F + t$ are disjoint, then by the lemma,

$$2|I| < \frac{4}{3} \cdot 2|F|.$$

$$2|I| = \frac{4}{3}|F \cup (F + t)|.$$

Steinhaus Theorem Proof (Continued)

Proof.

- By monotonicity,

$$2|I| \leq \frac{4}{3}|I \cup (I+t)| \leq \frac{4}{3}(|I|+|t|).$$

- Note that the equation does not hold for small $|t|$, thus F and $F+t$ intersect for small enough $|t|$, that is

$$|t| < \frac{1}{2}|I| \quad \Longrightarrow \quad F \cap (F+t) \neq \varnothing.$$

Thus, $F-F$ and $E-E$ must contain the interval $(\frac{-|I|}{2}, \frac{|I|}{2})$.

□

Proof of the existence of a Non-Measurable set

Theorem

The set N (defined earlier) is not Lebesgue measurable.

Proof.

- First we assume N is *measurable* for contradiction. Note that N contains exactly one element from each distinct equivalence class. Since these distinct equivalence classes partition $\mathbb{R}$,

$$\mathbb{R} = \bigcup_{x \in N} (\mathbb{Q} + x) = \bigcup_{x \in N} \bigcup_{q \in \mathbb{Q}} \{q + x\} = \bigcup_{q \in \mathbb{Q}} (N + q).$$

- As the external *Lebesgue measure* is translation invariant and has countable subadditivity,

$$\infty = |\mathbb{R}|_e = \left| \bigcup_{q \in \mathbb{Q}} (N + q) \right|_e \leq \sum_{q \in \mathbb{Q}} |N + q|_e = \sum_{q \in \mathbb{Q}} |N|_e.$$

Proof of the existence of a Non-measurable set (continued)

Theorem

The set N (defined earlier) is not Lebesgue measurable.

Proof.

- Thus $|N|_e > 0$ as $\infty \leq \sum_{q \in \mathbb{Q}} |N|_e$. Note that any two different points from N must come from two distinct equivalence classes of $\sim$ and must differ by an irrational value. $N - N$ contains no intervals; however, by *Steinhaus*, $N - N$ contains an interval, revealing the contradiction. Thus, N is not *Lebesgue measurable*.

□

Taking it further

Besides showing the existence of set that is not *Lebesgue measurable*, we can similarly prove that there is no measure function which satisfies our four "*nice*" properties of measure mentioned earlier.

Theorem

There exists no measure function $\mu : \mathcal{P}(\mathbb{R}) \to [0, \infty]$ *that satisfies all the properties (i-iv).*

- Using the equivalence classes of the relation $\sim$ over the interval [0, 1) and the **Axiom of Choice**, construct a set M.
- We assume that such a function exists for contradiction.
- We can create a union of countable (non-finite) distinct sets $M_k = M + q_k$ for $q_k \in \mathbb{Q} \cap [-1, 1]$. Moreover, we can bound its measure between two known measures,

$$1 = \mu([-1, 0)) \leq \mu(\bigcup_{k=1}^{\infty} M_k) \leq \mu([-1, 2)) = 3.$$

Taking it further (continued)

Theorem

There exists no measure function $\mu : \mathcal{P}(\mathbb{R}) \to [0, \infty]$ that satisfies all the properties (i-iv).

- By **countable additivity** and **translation invariance**,

$$\mu(\bigcup_{k=1}^{\infty} M_k) = \sum_{k=1}^{\infty} \mu(M_k) = \sum_{k=1}^{\infty} \mu(M).$$

- Since $\mu(M) \geq 0$, $\sum_{k=1}^{\infty} \mu(M) = 0$ if $\mu(M) = 0$ or $\sum_{k=1}^{\infty} \mu(M) = \infty$ if $\mu(M) > 0$. This contradicts,

$$1 \leq \mu(\bigcup_{k=1}^{\infty} M_k) = \sum_{k=1}^{\infty} \mu(M) \leq 3.$$

- Therefore, it is shown that no such function μ exists.

Thank you

Reference

Heil, C. (2019). *Introduction to Real Analysis.*Springer International Publishing

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